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An integer optimality condition for column generation on zero–one linear programs

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ABSTRACT

Column generation is a linear programming method that, when combined with appropriate integer programming techniques, has been successfully used for solving huge integer programs. The method alternates between a restricted master problem and a column generation subproblem. The latter step is founded on dual information from the former one; often an optimal dual solution to the linear programming relaxation of the restricted master problem is used.

We consider a zero–one linear programming problem that is approached by column generation and present a generic sufficient optimality condition for the restricted master problem to contain the columns required to find an integer optimal solution to the complete problem. The condition is based on dual information, but not necessarily on an optimal dual solution. It is however most natural to apply the condition in a situation when an optimal or near-optimal dual solution is at hand.

We relate our result to a few special cases from the literature, and make some suggestions regarding possible exploitation of the optimality condition in the construction of column generation methods for integer programs.

1. Introduction

Column generation is a method typically used for solving linear programming problems with a huge number of variables. When complemented with an appropriate integer programming technique, column generation has proved to be successful in solving certain classes of large scale integer programs.

The principle of linear programming column generation is to alternate between a restricted master problem and a column generation subproblem. The latter derives a new column for the restricted master problem, given the values of the dual variables of an optimal basis for the restricted master problem. The generation of columns continues until a best column has zero reduced cost, which is the well known optimality condition. With other words, all the columns needed for solving the full master problem are available in its restriction if all columns not included in the restriction have non-favourable reduced costs.

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For an integer program with a huge number of variables, the columns needed for solving its linear programming relaxation are typically not fully adequate for solving the integer program. In the well-known branch-and-price approach for solving integer programs, linear programming column generation is embedded in a branch-and-bound scheme, so that new columns can be generated at each node, as they are needed. For surveys of the column generation methodology and its applications within integer programming, see Lübbecke and Desrosiers [1], Barnhart et al. [2], Wilhelm [3], and Vanderbeck [4].

We are considering a fundamental question regarding zero–one linear programs that are approached by a column generation strategy: 'How do we know that a restricted master problem contains the columns needed to find an optimal integer solution to the complete master problem?' An answer lies in the generic integer optimality condition to be presented in Section 2. This condition has a structural similarity with the linear programming column generation optimality condition. Our condition is also based on dual information, which can be either linear programming or Lagrangian dual solutions, but in contrast to the traditional linear programming column generation optimality context, an optimal dual solution is not required.

The inspiration to our analysis is a column generation strategy studied in a paper by Sweeney and Murphy [5] from the late seventies. Although their computational results are now outdated, their column generation strategy is of principal interest in an integer programming setting. A similar strategy is a vital component of a method presented more recently by Baldacci et al. [6], where vehicle routing problems are successfully solved. Both these examples are described in Section 3, together with a third example from Baldacci et al. [7], which is very similar to the example from Baldacci et al. [6]. Further, we relate the optimality condition presented in Sweeney and Murphy [5] to a similar result presented in Larsson and Patriksson [8].

The earlier works are tailored to the particular structures of the problem formulations studied. The generic integer optimality condition presented in this paper provides a framework that suggests a wider range of potential applications of similar approaches and simplifies extensions to other settings.

1.1. Problem formulation and notation

The zero–one linear program to be solved by column generation is

$$[\mathrm{MP}_N] \qquad z_N^* = \min \sum_{j=1}^{N} c_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{N} a_j x_j = b, \quad (x_j)_{j=1}^N \in X \subseteq \{0,1\}^N.$$

This program is referred to as the complete master problem. Here, $c_j \in R$ and $a_j \in R^M$, $j = 1,\dots,N$, and $b \in R^M$. The set of columns is denoted by $\mathcal{P} = \{(c_j, a_j) : j = 1,\dots,N\} \subseteq R \times R^M$. This set is typically huge and each element is a feasible solution to an optimisation problem. In applications where columns can be identical apart from their costs, it is natural to include only the cheapest ones in $\mathcal{P}$. The set X is usually described by structural constraints, which are typically simple restrictions like convexity or cardinality constraints. The problem MP_N is assumed to be feasible.

We next assume that the variables with indices $j = 1,\dots,n$, where $n < N$, are at hand. Then a corresponding restricted zero–one program, denoted by MP_n and referred to as the restricted master problem, is obtained. Typically $n \ll N$ holds. The problem MP_n is assumed to be feasible. Let its optimal objective value be z_n^*. Let $\bar{z}$ denote the objective value of a feasible zero–one solution to MP_n; such a solution clearly also provides a feasible solution and an upper bound to MP_N. (Throughout the paper we use bars to indicate upper and lower bounds on various objective value notations.)

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Let $u \in R^M$ be a vector of dual multipliers associated with the linear equality constraints of problem MP_N. Further, let

$$\bar{c}_j = c_j - u^{\mathrm{T}} a_j, \quad j = 1, \dots, N,$$

be the corresponding Lagrangian reduced costs, as obtained if a Lagrangian relaxation of the linear equality constraints is made. The column generation subproblem then becomes

$$[\mathrm{CG}] \qquad \bar{c} \;=\; \begin{array}{ll} \min & c - u^{\mathrm{T}} a \\ \text{s.t.} & (c, a) \in \mathcal{P}. \end{array}$$

Remark 1 (*Choice of Dual Multiplier Values*)**.** The column generation subproblem CG is typically solved for an *optimal* dual solution to the linear programming relaxation of MP_n, although this is not always the case. Examples of methods working with possibly non-optimal dual solutions are when: stabilised column generation is used, see Section 6.2 of Lübbecke and Desrosiers [1]; the linear programming relaxation of the restricted master problem is solved by Lagrangian relaxation and dual subgradient optimisation, see, e.g., Huisman et al. [9]; the linear programming relaxation of the restricted master problem is solved by a dual analytic centre approach, see, e.g., Goffin et al. [10]; the integer restricted master problem is approached by an integral simplex method, see Rönnberg and Larsson [11,12]. Our analysis allows *any* choice of dual multiplier values, including optimal and near-optimal dual solutions obtained from linear programming or Lagrangian relaxation. ■

2. Optimality condition

The optimality condition to be derived is sufficient for a restricted master problem MP_n to contain an optimal solution to the complete master problem MP_N. This optimality condition generalises the well known linear programming optimality condition that, given an optimal dual solution to the restricted master problem, the restriction contains all the columns needed for solving the full problem if the reduced costs of all the columns not included in the restriction are non-negative.

Our condition requires dual multiplier values (not necessarily from a dual optimal solution of the linear programming relaxation of the restricted master problem), the best Lagrangian reduced cost among the columns not yet included in MP_n, a lower bound on the optimal value of MP_n, and an upper bound on the optimal value of MP_N. The best possible lower bound for MP_n is of course z_n^*, but this value can be expensive to compute. The lower bound can alternatively be found by solving a problem that is significantly easier than MP_n, or obtained as a by-product of other calculations.

2.1. Lower bounding

As was introduced for MP_N, X is a subset of $\{0,1\}^N = \{0,1\}^n \times \{0,1\}^{N-n}$, that is, the Cartesian product of the possible values of the variables in MP_n and the possible values of the variables for the columns not yet generated. Let $X_n \subseteq \{0,1\}^n$ be the projection of X onto $\{0,1\}^n \times \{0\}^{N-n}$, and let $X_{N-n} \subseteq \{0,1\}^{N-n}$ be the projection of X onto $\{0\}^n \times \{0,1\}^{N-n}$. Let the set $Y_n \subseteq [0,1]^n$ be such that $X_n \subseteq Y_n$ holds, and note that it then holds that $X \subseteq Y_n \times \{0,1\}^{N-n}$. (For example, in case the set X_n is described by linear constraints and integrality conditions, the set Y_n can be chosen as the continuous relaxation of X_n.) These sets and their relationships are illustrated in Figs. 1–3.

For the optimal objective value of MP_n, a lower bound to be used in the optimality condition is introduced in the following lemma, where the set Y_n plays an important role.

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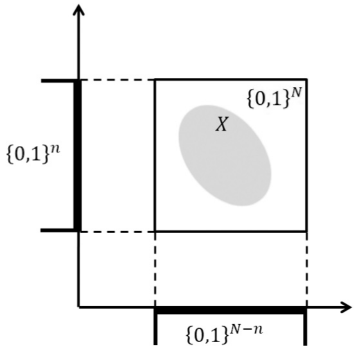

Fig. 1. The set X is a subset of $\{0,1\}^N$, which is the Cartesian product of the sets $\{0,1\}^n$, illustrated on the vertical axis, and $\{0,1\}^{N-n}$, illustrated on the horizontal axis.

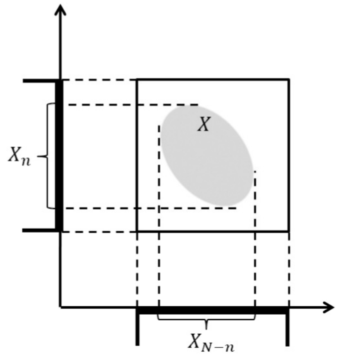

Fig. 2. The set X_n is the projection of X onto $\{0,1\}^n \times \{0\}^{N-n}$, while X_{N-n} is the projection of X onto $\{0\}^n \times \{0,1\}^{N-n}$.

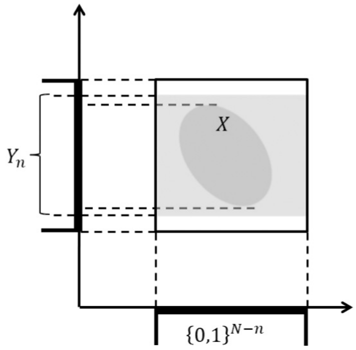

Fig. 3. The set X is contained in the Cartesian product of the sets Y_n and $\{0,1\}^{N-n}$.

Lemma 1. *Let z_n^* be the optimal value of problem* MP_n, $X_n \subseteq Y_n \subseteq [0,1]^n$, *and* $u \in R^M$. *Define*

$$\underline{z}_n = u^{\mathrm{T}} b + \min\left\{ \sum_{j=1}^{n} \bar{c}_j x_j \ : (x_j)_{j=1}^n \in Y_n \right\}.$$

Then $\underline{z}_n \leq z_n^*$ *holds.*

Proof.

$$\begin{aligned} z_n^* &= \min\left\{\sum_{j=1}^n c_j x_j \;:\; \sum_{j=1}^n a_j x_j = b; (x_j)_{j=1}^n \in X_n\right\} \\ &= u^{\mathrm{T}} b + \min\left\{\sum_{j=1}^n \bar{c}_j x_j \;:\; \sum_{j=1}^n a_j x_j = b; (x_j)_{j=1}^n \in X_n\right\} \\ &\geq u^{\mathrm{T}} b + \min\left\{\sum_{j=1}^n \bar{c}_j x_j \;: (x_j)_{j=1}^n \in Y_n\right\} \\ &= \underline{z}_n \end{aligned}$$

The second equality follows from adding the zero term $u^{\mathrm{T}}\left(b - \sum_{j=1}^n a_j x_j\right)$ to the objective function, while the inequality holds because the feasible set is enlarged by relaxing the linear constraints and replacing X_n by Y_n. ■

Remark 2 (*Quality of the Lower Bound*)**.** With the choice $Y_n = X_n$, the value $\underline{z}_n$ is a Lagrangian relaxation lower bound for MP_n.

If (i) the set X is described by linear constraints and zero–one restrictions, (ii) the set Y_n is chosen as the continuous relaxation of X_n obtained by dropping the zero–one restrictions, and (iii) u is an optimal dual solution for the linear programming relaxation of MP_n, then it holds that $\underline{z}_n$ is the optimal value of the linear programming relaxation of MP_n.

Further, if the continuous relaxation of the set X has the integrality property (see e.g. Chapter II.3.6 in Nemhauser and Wolsey [13]) then the linear programming bound and the best possible Lagrangian relaxation bound coincide. In case the integrality property does not hold, then the best possible Lagrangian bound can be expected to be stronger than the linear programming bound. ■

Remark 3 (*A Special Case of Lower Bound*)**.** If (i) the set X is described by linear constraints and zero–one restrictions, (ii) the set Y_n is chosen as the continuous relaxation of X_n, (iii) problem MP_n has been shown to contain the columns necessary to find an optimal solution to the linear programming relaxation of MP_N, (iv) u is an optimal dual solution for the linear programming relaxation of MP_n (and then also of MP_N), and (v) Y_N is the continuous relaxation of X, then it holds that $\underline{z}_n$ coincides with

$$\underline{z}_N = u^{\mathrm{T}} b + \min\left\{\sum_{j=1}^N \bar{c}_j x_j \;: (x_j)_{j=1}^N \in Y_N\right\},$$

which in turn coincides with the optimal value of the linear programming relaxation of MP_N. ■

2.2. Conditional lower bounding

The optimality condition to be presented takes into account the fact that, due to the constraints describing the set X, adding a new variable to the restricted master problem and letting it take the value one can influence the feasible values of the variables already included in the master problem.

We assume that the set of indices of the variables not included in the restricted master problem consists of subsets $J_{N-n}^s \subseteq \{n+1,\ldots,N\}$, $s = 1,\ldots,S$, such that $\cup_{s=1,\ldots,S} J_{N-n}^s = \{n+1,\ldots,N\}$. For each $s = 1,\ldots,S$, let $k_s \in J_{N-n}^s$ be such that

$$\bar{c}_{k_s} = \min\left\{\bar{c}_j \;:\; j \in J_{N-n}^s\right\}, \tag{1}$$

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so that the corresponding variable, x_{k_s}, is a best variable from the subset J^s_{N-n} with respect to Lagrangian reduced cost.

Property 1 (*Restriction Property*). *Assume that, whenever at least one of the variables with an index in a set J^s_{N-n}, $s = 1, \ldots, S$, takes the value one, then a restriction on the variables in* MP_n *is imposed, that is,*

$$(x_j)_{j=1}^N \in X \text{ and } \sum_{j \in J^s_{N-n}} x_j \geq 1 \Rightarrow$$
$$(x_j)_{j=1}^n \in Y_n \cap X^{R_s} \subseteq [0,1]^n, \quad s = 1, \ldots, S, \tag{2}$$

where $X^{R_s} \subseteq [0,1]^n$ is the set that imposes the restriction on MP_n.

Given such restrictions, define corresponding conditional lower bounds

$$\underline{z}_{R_s} = u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^n \bar{c}_j x_j \ : (x_j)_{j=1}^n \in Y_n \cap X^{R_s} \right\}, \quad s = 1, \ldots, S. \tag{3}$$

2.3. Main result

The following theorem is our main result. It gives a sufficient condition for the restricted master problem to contain enough columns to provide an optimal solution to the complete master problem.

Theorem 1. *Let z^*_N and z^*_n be the optimal values of problems MP_N and MP_n, respectively, let $\bar{z}$ be the objective value of a feasible solution to MP_n, let $\underline{z}_{R_s}$ be defined by expression (3), and let $\bar{c}_{k_s}$, $s = 1, \ldots, S$, be defined by expression (1). If*

$$\bar{c}_{k_s} \geq \max\{0, \bar{z} - \underline{z}_{R_s}\} \quad s = 1, \ldots, S, \tag{4}$$

holds, then $z^*_N = z^*_n$.

Proof. For each $s = 1, \ldots, S$, study the optimal objective value of the restriction of MP_N obtained when $\sum_{j \in J^s_{N-n}} x_j \geq 1$ must hold:

$$\min \left\{ \sum_{j=1}^N c_j x_j \ : \ \sum_{j=1}^N a_j x_j = b; \ \sum_{j \in J^s_{N-n}} x_j \geq 1; (x_j)_{j=1}^N \in X \right\} =$$
$$u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^N \bar{c}_j x_j \ : \ \sum_{j=1}^N a_j x_j = b; \ \sum_{j \in J^s_{N-n}} x_j \geq 1; (x_j)_{j=1}^N \in X \right\} \geq$$
$$u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^N \bar{c}_j x_j \ : \ \sum_{j \in J^s_{N-n}} x_j \geq 1; (x_j)_{j=1}^N \in X \right\} \geq$$
$$u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^N \bar{c}_j x_j \ : \ \sum_{j \in J^s_{N-n}} x_j \geq 1; (x_j)_{j=1}^N \in \left(X_n \cap X^{R_s}\right) \times \{0,1\}^{N-n} \right\} =$$
$$u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^n \bar{c}_j x_j + \bar{c}_{k_s} \ : (x_j)_{j=1}^n \in X_n \cap X^{R_s} \right\} \geq$$

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$$u^{\mathrm{T}}b + \min\left\{\sum_{j=1}^{n} \bar{c}_j x_j \; : (x_j)_{j=1}^{n} \in Y_n \cap X^{R_s}\right\} + \bar{c}_{k_s} =$$
$$\underline{z}_{R_s} + \bar{c}_{k_s} \geq \bar{z} \geq z_n^*.$$

The second inequality follows from $(x_j)_{j=1}^{N} \in X \subseteq X_n \times \{0,1\}^{N-n}$ and the implication (2). The second equality holds because $\bar{c}_j \geq 0$, $j \in J_{N-n}^s$, $s = 1, \ldots, S$, which in turn follows from (1) and (4). The fourth inequality is implied by (4).

Hence, for each $s = 1 \ldots, S$ it holds that no feasible solution to MP_N with $\sum_{j \in J_{N-n}^s} x_j \geq 1$ can be better than an optimal solution to MP_n. By also using that $\cup_{s=1,\ldots,S} J_{N-n}^s = \{n+1, \ldots, N\}$ holds, it is concluded that $z_N^* = z_n^*$. ■

The conclusion that $z_N^* = z_n^*$ holds means that the restricted master problem contains enough columns to solve the complete master problem. Note however that an optimal solution to the restricted master problem remains to be found.

Remark 4 (*Special Cases of Theorem 1*). The practical use of the optimality condition of Theorem 1 depends on the structure of the problem and how the subsets J_{N-n}^s, $s = 1, \ldots, S$, can be formed. Cases of special interest are when $S = 1$, see the application in Section 3.2, and when the subsets J_{N-n}^s, $s = 1, \ldots, S$ form a partition of the variables, see the application in Section 3.3.

The sets J_{N-n}^s, $s = 1, \ldots, S$, do however not need to constitute a partitioning of the set $\{n+1, \ldots, N\}$. An example of this is if MP_N is a set partitioning problem (see, e.g., Barnhart et al. [2]) and a set J_{N-n}^s is introduced for each row of the constraint matrix. ■

The applications presented in Section 3 illustrate how the restriction property can be expressed and utilised. One way to measure its impact is to compare the lower bounds $\underline{z}_{R_s}$ and $\underline{z}_n$, from Eq. (3) and Lemma 1, respectively. For this purpose we introduce

$$\Delta_s = \underline{z}_{R_s} - \underline{z}_n, \quad s = 1, \ldots, S. \tag{5}$$

The requirement $\bar{c}_{k_s} \geq \bar{z} - \underline{z}_{R_s}$, $s = 1, \ldots, S$, that follows from (4), can then alternatively be expressed as

$$\bar{c}_{k_s} + \Delta_s \geq \bar{z} - \underline{z}_n, \quad s = 1, \ldots, S. \tag{6}$$

Since, $\underline{z}_{R_s} \geq \underline{z}_n$ holds, so does $\Delta_s \geq 0$, $s = 1, \ldots, S$. In the case when $Y_n \cap X^{R_s} \subset Y_n$, there is a chance that $\underline{z}_{R_s} > \underline{z}_n$, so that $\Delta_s > 0$, holds. If it is not known a priori which effect the restrictions $\sum_{j \in J_{N-n}^s} x_j \geq 1$, $s = 1, \ldots, S$, have with respect to the possible values of the variables already at hand in MP_n, then $\underline{z}_{R_s} = \underline{z}_n$ and $\Delta_s = 0$, $s = 1, \ldots, S$, hold. In this case, the following weaker sufficient condition is obtained.

Corollary 2. *Let z_N^* and z_n^* be the optimal values of problems MP_N and MP_n, respectively, let $\bar{z}$ be the objective value of a feasible solution to MP_n, let $\underline{z}_n$ be a lower bound on z_n^* obtained according to Lemma 1, and let $\bar{c}_{k_s}$, $s = 1, \ldots, S$, be defined by expression (1). If*

$$\bar{c}_{k_s} \geq \bar{z} - \underline{z}_n, \quad s = 1, \ldots, S, \tag{7}$$

holds, then $z_N^* = z_n^*$.

Under additional assumptions, the result of Corollary 2 reduces into a known integer optimality condition, as pointed out in the remark below.

Remark 5 (*Special case of Corollary 2*). If (i) $X_n = \{0,1\}^n$, $Y_n = [0,1]^n$, MP_n has been shown to contain the columns needed to find an optimal solution to the linear programming relaxation of MP_N, (ii) u is an optimal dual solution for the linear programming relaxation of MP_n (and then also of MP_N), and (iii) $\bar{z} = z_n^*$, then it holds that $\underline{z}_n$ coincides with the optimal value of the linear programming relaxation of MP_N, $\underline{z}_N$ (cf. Remark 3 for the case $Y_N = [0,1]^n$), and the sufficient condition (7) for $z_N^* = z_n^*$ to hold reduces to

$$\bar{c}_j \geq z_n^* - \underline{z}_N, \quad j = n+1, \ldots, N,$$

with $\bar{c}_j$ now being the ordinary linear programming reduced cost. This condition simply states that the linear programming reduced costs of the columns not included in MP_n should be at least as large as the duality gap of MP_N. This result can of course also be easily derived by applying arguments from linear programming sensitivity analysis. ■

The optimality condition is designed to be used in a method where columns are generated using dual solutions which are not necessarily optimal with respect to the linear programming relaxation of MP_n. Such methods can be designed such that only bounds on the quantities of condition (4) ($\bar{z}$, $\underline{z}_{R_s}$, and $\bar{c}_{k_s}$, $s = 1, \ldots, S$) and (7) ($\bar{z}$, $\underline{z}_n$, and $\bar{c}_{k_s}$, $s = 1, \ldots, S$) are used.

3. Examples of applications

We have found in the literature two implemented solution strategies, by Sweeney and Murphy [5] and by Baldacci et al. [7,6], that utilise optimality conditions that can be considered to be problem tailored versions of the condition of Theorem 1. In this section, we show how these tailored conditions can be reproduced and in some cases also strengthened by applying our new condition.

3.1. Solution strategy

An outline of the kind of solution strategy proposed by Sweeney and Murphy [5] is given below. The solution strategies in Baldacci et al. [7,6] are essentially the same as the one in Sweeney and Murphy [5], but derived independently. The strategy in Baldacci et al. [6] differs slightly in the respect that it goes through the steps below only once.

1. Find some values for the dual multipliers u. For example, one can use an optimal dual solution of the linear programming relaxation of MP_n.
2. For each subproblem, enumerate a number of the best solutions with respect to the Lagrangian reduced costs, and include these in MP_n. Calculate the lower bound $\underline{z}_n$ using u.
3. Solve MP_n and obtain $\bar{z}$.
4. Check the optimality condition (in terms of values of reduced costs of columns not generated, see [5–7]). If it is fulfilled, then terminate, otherwise continue.
5. From the subproblems that provide improving columns, generate more columns, and go to Step 3.

3.2. Sweeney and Murphy [5]

The results of Sweeney and Murphy [5] apply to a setting with multiple column generation subproblems and where the master problem has convexity constraints stating that exactly one column from each subproblem shall be used. The application studied is a production planning problem that in the late seventies

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was considered to be large-scale. It is shown below that the optimality condition presented and proved by Sweeney and Murphy [5] is a special case of the condition of Theorem 1, obtained when the convexity constraints induce a partition of the variables not included in the restricted master problem.

Sweeney and Murphy [5] remark that the best possible choice of dual values in their optimality condition is one that is optimal in the linear programming relaxation of MP_N. Such a dual solution is not easily available, and they therefore consider different options for finding good approximations thereof. They also discuss a computational trade-off between the burden of finding good dual values and the burden of enumerating enough subproblem solutions to find an optimal solution to the master problem.

The optimality condition of Sweeney and Murphy [5] requires knowledge of which columns that are included in which of the convexity constraints. We capture this by indexing convexity constraints and subproblems with $s = 1, \ldots, S$, and letting J_N^s denote the set of indices for the columns that are included in convexity constraint s. Further, use the notation J_n^s for the indices of the columns in convexity constraint s that are included in problem MP_n, and J_{N-n}^s for those columns that are not, $s = 1, \ldots, S$.

When the model in Sweeney and Murphy [5] is described using our notation for the master problem, the objective function remains the same, and so do the linear constraints if the equalities are changed into inequalities. To derive a condition similar to the one used by Sweeney and Murphy [5], we choose the set X as

$$X = \left\{ (x_j)_{j=1}^N \in \{0,1\}^N : \sum_{j \in J_N^s} x_j = 1, \; s = 1, \ldots, S \right\}.$$

The lower bound to be used in (6) can be derived as follows, with $Y_n = X_n$.

$$\begin{aligned} \underline{z}_n &= u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^n \bar{c}_j x_j : \sum_{j \in J_n^s} x_j = 1, \; s = 1, \ldots, S; \; (x_j)_{j=1}^n \in \{0,1\}^n \right\} = \\ &= u^{\mathrm{T}} b + \sum_{s=1}^S \min_{j \in J_n^s} \bar{c}_j \end{aligned}$$

For each subproblem $s = 1, \ldots, S$, the requirement $\sum_{j \in J_{N-n}^s} x_j \geq 1$ here implies that all variables from the same subproblem s that are included in the restricted master problem must take the value zero. Hence,

$$X^{R_s} = \left\{ (x_j)_{j=1}^n \in \{0,1\}^n : \; x_j = 0, \; j \in J_n^s \right\}.$$

The remaining quantities to be used in (6) thus become

$$\underline{z}_{R_s} = u^{\mathrm{T}} b + \sum_{r \in \{1,\ldots,S\} \setminus \{s\}} \min_{j \in J_n^r} \bar{c}_j, \quad s = 1, \ldots, S,$$

and

$$\Delta_s = \underline{z}_{R_s} - \underline{z}_n = - \min_{j \in J_n^s} \bar{c}_j, \quad s = 1, \ldots, S,$$

which gives the optimality condition

$$\bar{c}_{k_s} - \min_{j \in J_n^s} \bar{c}_j \geq \bar{z} - \underline{z}_n, \quad s = 1, \ldots, S. \tag{8}$$

One interpretation of this condition is that it is not only the reduced cost of the variable entering MP_n that matters, but also the reduced cost of the variable from the same subproblem as x_k, that will no longer be used. Since this variable is not known, an appropriate bound is obtained by assuming it is the best variable from the same subproblem.

A slight difference between our Theorem 1 and the corresponding optimality condition in Sweeney and Murphy [5] is that instead of using $\bar{c}_{k_s} = \min\left\{\bar{c}_j \,:\, j \in J^s_{N-n}\right\}$, which is the reduced cost of the best column not included in MP$_n$, they use the value $\bar{c}_{k_s} = \max\left\{\bar{c}_j \,:\, j \in J^s_n\right\}$, which is the reduced cost of the worst column already included in MP$_n$. In Sweeney and Murphy [5] columns are ranked based on increasing reduced costs and added to MP$_n$ in this order. Their way of constructing MP$_n$ implies that

$$\max\left\{\bar{c}_j \,:\, j \in J^s_n\right\} \le \min\left\{\bar{c}_j \,:\, j \in J^s_{N-n}\right\} \tag{9}$$

holds. Hence our Theorem 1 implies their optimality condition.

The derivation of the optimality condition of Sweeney and Murphy [5] relies heavily on the presence of convexity constraints, and the result can therefore not be extended beyond this class of problems without using some other argumentation. A similar optimality condition, in the presence of convexity constraints, is presented in Larsson and Patriksson [8]; its relationship to the result of Sweeney and Murphy [5] is commented on in the following remark.

Remark 6. An optimality condition similar to (8) is given in Larsson and Patriksson [8, Prop. 15, case (b)]. It is stated for a problem that contains linear inequality constraints and several convexity constraints, and considers arbitrary dual multipliers. For the case of equality constraints and with our notations their condition becomes

$$\bar{c}_k - \min_{j \in J^s_N} \bar{c}_j \ge \bar{z} - \underline{z}_N, \quad k \in J^s_{N-n}, \quad s = 1, \ldots, S,$$

where $\underline{z}_N$ is a Lagrangian relaxation lower bound obtained for the dual multipliers at hand and given by

$$\underline{z}_N = u^{\mathrm{T}} b + \sum_{s=1}^{S} \min_{j \in J^s_N} \bar{c}_j.$$

Given that condition (9) holds, this optimality condition becomes

$$\bar{c}_k - \min_{j \in J^s_n} \bar{c}_j \ge \bar{z} - \left(u^{\mathrm{T}} b + \sum_{s=1}^{S} \min_{j \in J^s_n} \bar{c}_j \right), \quad k \in J^s_{N-n}, \quad s = 1, \ldots, S,$$

which coincides with condition (8) if $\underline{z}_n$ is chosen as a Lagrangian relaxation lower bound for MP$_n$.

3.3. Baldacci et al.

In both the works by Baldacci et al. [7,6], the master problem to be solved is a set partitioning problem with a cardinality constraint on the number of columns to be used in the solution. In the respective papers, the master problem originates from different applications; in Baldacci et al. [7] the original problem to be solved is a multiple disposal facilities and multiple inventory locations rollon–rolloff vehicle routing problem, and in Baldacci et al. [6] it is a capacitated vehicle routing problem.

In both papers, an important contribution is a heuristic approach for efficiently finding a near-optimal dual solution to the linear programming relaxation of the master problem. In Baldacci et al. [6], this heuristic is combined with adding cuts to the linear programming relaxation of the master problem in order to strengthen the formulation and obtain tighter bounds. The heuristic approach used in both Baldacci et al. [7] and Baldacci et al. [6], together with the cuts in Baldacci et al. [6], are key ingredients for obtaining the excellent computational results that they report.

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The optimality condition used by Baldacci et al. [7] is related to Corollary 2 and its validity is motivated by linear programming results. The optimality condition used can, however, be strengthened by using the result of Theorem 1, and it is shown below how this can be achieved.

Comparing the optimality conditions used in Baldacci et al. [7,6] to condition (7) in Corollary 2, the former use a lower bound on $\underline{z}_n$ rather than the exact value and they also use the formulation with $\max\{\bar{c}_j \ : \ j = 1, \dots, n\}$ instead of $\min\{\bar{c}_j \ : \ j = n+1, \dots, N\}$. To derive the results found in these papers within our setting, the model MP_N is formulated with the set partitioning constraints, the cuts, and the cardinality constraints as explicit linear constraints, and with $X = \{0,1\}^N$. These choices yield the optimality condition

$$\max_{j=1,\dots,n} \bar{c}_j \geq \bar{z} - \underline{z}_n.$$

The reason why the same optimality conditions can be obtained irrespective of using our result or their linear programming based argument is that in our case, the choice $X = \{0,1\}^N$ gives a subproblem with the integrality property and, as is well known from the literature (see e.g. Chapter II.3.6 in Nemhauser and Wolsey [13]), the optimal values of the Lagrangian dual problem and the linear programming dual problem are the same. We next show how to formulate the condition of Theorem 1 for the setting in Baldacci et al. [7,6].

In order to obtain the condition of Theorem 1 for the problems solved in those papers, we choose to let the set X be defined by the cardinality constraint, such that

$$X = \left\{ (x_j)_{j=1}^N \in \{0,1\}^N : \sum_{j=1}^N x_j \leq L \right\},$$

and we treat the rest of the linear constraints explicitly in MP_N. To simplify the presentation below we assume, without loss of generality, that the variables of MP_n are indexed such that $\bar{c}_1 \leq \bar{c}_2 \leq \cdots \leq \bar{c}_n$.

Using $Y_n = X_n$, and assuming that $n \geq L$, the lower bound becomes

$$\begin{aligned} \underline{z}_n &= u^{\mathrm{T}} b + \min \left\{ \sum_{j=1}^n \bar{c}_j x_j \ : \ \sum_{j=1}^n x_j \leq L; (x_j)_{j=1}^n \in \{0,1\}^n \right\} = \\ &= u^{\mathrm{T}} b + \sum_{j=1}^L \min\{0, \bar{c}_j\}. \end{aligned}$$

For this application $S = 1$ and we therefore drop the index s in the following and use the notation $J_{N-n} = \{n+1, \dots, N\}$ for the single subset of indices. The restriction on MP_n imposed by $\sum_{j \in J_{N-n}} x_j \geq 1$ is

$$X^R = \left\{ (x_j)_{j=1}^n \in \{0,1\}^n : \sum_{j=1}^n x_j \leq L - 1 \right\},$$

which gives

$$\underline{z}_R = u^{\mathrm{T}} b + \sum_{j=1}^{L-1} \min\{0, \bar{c}_j\}$$

and

$$\varDelta = \underline{z}_R - \underline{z}_n = -\min\{0, \bar{c}_L\},$$

to be used in Theorem 1. Condition (6) hence becomes

$$\bar{c}_k - \min\{0, \bar{c}_L\} \geq \bar{z} - \underline{z}_n.$$

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An interpretation of this condition is that if the solution that corresponds to the lower bound $\underline{z}_n$ contains L variables with the value one, and a new variable enters the restricted master problem with the value one, then a pessimistic bound on the change in objective value is the difference between the reduced cost of the new variable x_k and the reduced cost of x_L, which is the least favourable one among the variables currently taking the value one in the restricted master problem. Hence, the difference between the condition of Corollary 2 and this one is to take into account that a new variable replaces an old one in the cardinality constraint.

4. Practical use

As is well known, the columns that are required to solve the linear programming relaxation of a complete master problem may not be very useful for solving the integer program, since the information contained in the linear programming optimal dual solutions will not necessarily be a reliable guide to which columns that are useful for the latter purpose.

When the goal is to solve the integer program, the intuition suggests that it can be beneficial to augment the restricted master problem with not only single columns of minimal reduced costs, but with several columns having small reduced costs. An enumeration of such columns, or rather of solutions to column generation subproblems CG, is a vital component in Sweeney and Murphy [5], Baldacci et al. [7,6], as well as in Rönnberg and Larsson [12], where it is referred to as over-generation of columns. An enumeration strategy can be computationally profitable for applications where a set of near-optimal subproblem solutions can be generated efficiently. An example of a well-known optimisation problem that allows efficient enumeration of a number of near-optimal solutions is the shortest path problem; see, e.g., Hadjiconstantinou and Christofides [14].

The strategy of enumerating columns with lowest reduced costs can be used, in principle, to produce enough columns to comply with the bound given by the optimality condition of Theorem 1 or that of Corollary 2; such an enumeration will guarantee that the restricted problem contains an optimal solution to the complete master problem. Column enumeration with the aim of solving the integer program can be made for any dual multipliers found in the column generation process, but because of the nature of the optimality condition it appears preferable to base it on dual multiplier values that are (near-)optimal in the dual of the linear programming relaxation of MP_N, and further let $\bar{z}$ be given by a (near-)optimal solution to MP_n.

A column enumeration strategy may of course also be useful in the process of solving the linear programming relaxation of the complete master problem, as a means for accelerating the column generation (and at the same time providing more columns that can be beneficial in the integer program).

The dual multiplier values used for column generation typically originate from solving the linear programming relaxation of the restricted master problem or its dual, but other choices of dual multipliers can also be quite reasonable. The approaches in Baldacci et al. [7,6] are examples of the use of dual multiplier values that have other origins than a linear programming relaxation. Another example of a column generation strategy where the dual multipliers also have another origin is the all-integer pivoting setting of Rönnberg and Larsson [11,12], where a linear programming optimal basis for the restricted master problem is in general not available.

Important to notice is also that whenever the dual multiplier values used in the column generation subproblem are not optimal with respect to the linear programming relaxation of the current restricted master problem, then an optimal solution to the subproblem might very well be a column which is already present in the restricted master problem, and adding this column does of course not give any progress. This risk must be taken into account in the design of a method.

An immediate way to ensure progress is to request the column generation subproblem to provide a column which is not already present in the restricted master problem, and this can be accomplished by using enumeration of near-optimal subproblem solutions.

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In this context, it can be mentioned that the principle of enumerating columns, with respect to reduced costs or some other measure, is sometimes also used as a pragmatic means for initiating a column generation scheme or for accelerating it.

We also note that enumeration strategies have been successfully applied to certain discrete optimisation problems in a Lagrangian relaxation framework. Handler and Zang [15] utilised a ranking methodology based on the Lagrangian reduced costs to solve a knapsack-constrained shortest-path problem. A more recent contribution to this methodology is found in Carlyle et al. [16]. In this application, the Lagrangian relaxation gives rise to a shortest path problem, for which near-optimal solutions are enumerated until an optimal solution to the original problem has been found.

Such a procedure is analogous to enumerating near-optimal columns in an application where the set X is defined by *one* convexity constraint together with zero–one restrictions.

Since the role of the integer restricted master problem is to combine the available columns to find a solution, it should benefit from a structure where the number of feasible solutions grows quickly thanks to enumeration. This is the case when the variables of MP_N are partitioned in groups and each of these should comply with a convexity constraint, that is, the set X is a non-trivial Cartesian product. Then the columns can be enumerated separately for each of the individual product sets. Because of the product structure, the number of ways to combine these columns will grow very quickly with the number of columns in each individual set.

5. Conclusions

The contribution of this paper is the presentation of a generic optimality condition for zero–one linear programming column generation that generalises, and is at least as strong as, optimality conditions previously known from the literature. In the papers referred to, the optimality conditions are an important component in the design of efficient solution methods for their respective applications. They are however tailored to the special structure of the problem formulation at hand, and can therefore not be extended directly to other problem formulations or applications. The results of this paper simplify such extensions.

As outlined in Section 4, algorithmic strategies based on enumeration of subproblem solutions can be used for several purposes. In particular, the principle of enumerating subproblem solutions according to increasing costs has in a Lagrangian relaxation context shown to be successful for some applications. These applications appear to have in common that there is only one Lagrangian subproblem (that is, the Lagrangian relaxation does not induce any separation), and that the enumeration then directly acts as a search for an optimal solution to the original problem. We believe that the principle of enumerating subproblem solutions has a greater potential for being useful in the column generation context, whenever there are several column generation subproblems (that is, the set X is a non-trivial Cartesian product). The reason for this is that, in contrast to the Lagrangian relaxation context described above, the restricted master problem then provides the option of combining all the available columns in the best possible way, with respect to overall feasibility and objective value. Our belief in the usefulness of enumeration strategies in the column generation context is supported by the results obtained by Sweeney and Murphy [5] and Baldacci et al. [7,6], and we consider this line of research to be promising. The general optimality condition that we present is at least as strong as those in Sweeney and Murphy [5] and Baldacci et al. [7,6], it is however not known if this difference in strength would be of importance in their applications.

The generality of the optimality condition presented in this paper provides a platform for the development of column generation methods for zero–one programs based on column enumeration and on the use of dual multiplier values that can be obtained by other means than solving a linear programming relaxation of the restricted master problem. Examples of existing such settings are those in Rönnberg and Larsson [12], Sweeney and Murphy [5] and Baldacci et al. [7,6]. We believe it to be an interesting area of future research to study more applications and problem structures.

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